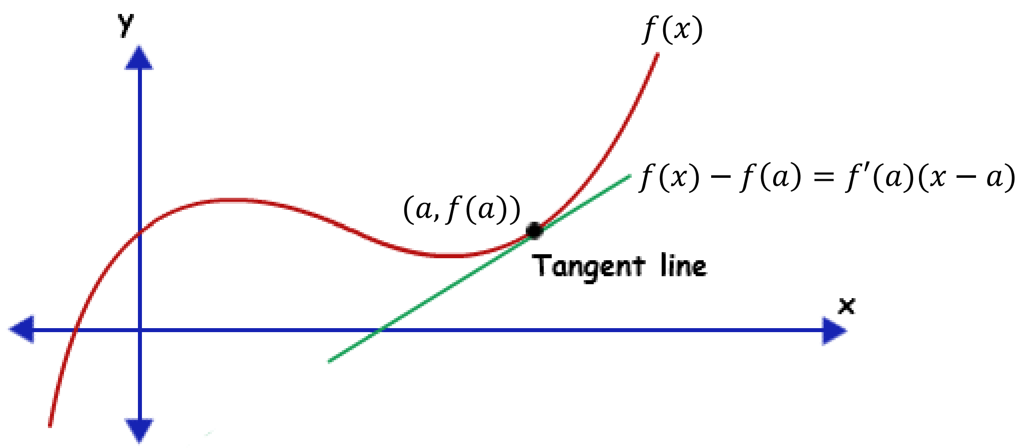
Tangents and Normals

Equation of a Tangent Line

The derivative at a point x = a, denoted $f'(a)$, is the instantaneous rate of change at that point. Geometrically, $\boldsymbol{f'(a)}$ gives us the **slope of the tangent line at** the point **x = a**.



Recall: A **tangent line** is a line that "just touches" a curve at a specific point without intersecting it.

To find the **equation of the tangent line** we need its **slope** and a **point** on the line. Given the function $f(x)$ and the point $(a, f(a))$ we can find the equation of the tangent line using the slope equation.

$$\mathbf{m} = \frac{f(x) - f(a)}{x - a}$$

Since $\boldsymbol{f'(a)}$ **gives us the slope of the tangent line** at the point **x = a,** we have

$$\mathbf{f'(a)} = \frac{f(x) - f(a)}{x - a}$$

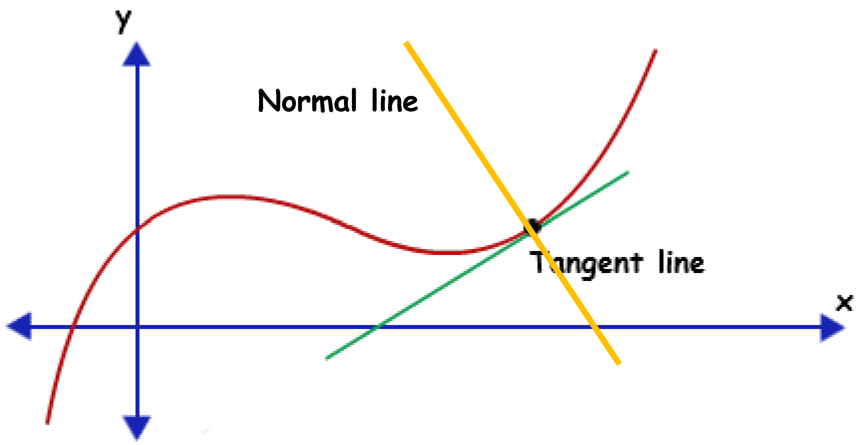
As such, the equation of the tangent line at x = a can be expressed as:

$$f(x) - f(a) = \mathbf{f'(a)}(x - a)$$

Equation of a Normal Line

The **normal line** is defined as the line that is perpendicular to the tangent line at the point of tangency. Knowing this, we can find the equation of the normal line at x = a by taking the **negative inverse of the slope** of the tangent line equation.

Thus, if $\boldsymbol{f'(a)}$ is the slope of the tangent line at x = a. The **negative inverse** is $\frac{\mathbf{-1}}{\mathbf{f'(a)}}$.



As such, the equation of the normal line at x = a can be expressed as:

$$f(x) - f(a) = \frac{\mathbf{-1}}{\mathbf{f'(a)}}(x - a)$$

Example 1: Find the equation of the tangent and normal lines of the function $f(x) = \sqrt{2x-1}$ at the point (5, 3).

Solution:

a) Equation of the Tangent Line.

Step 1: Find the slope of the function by solving for its first derivative.	$f(x) = \sqrt{2x-1}$ $f(x) = (2x-1)^{\frac{1}{2}}$ $f'(x) = \frac{1}{2}(2x-1)^{\frac{-1}{2}}(2)$ $f'(x) = \frac{1}{\sqrt{2x-1}}$

Step 2: Knowing $f'(a)$, solve for the slope of the tangent at $a = 5$.	$f'(5) = \frac{1}{\sqrt{2(5)-1}}$ $f'(5) = \frac{1}{3}$
Step 3: Solve for $f(a)$.	$f(5) = \sqrt{2(5)-1}$ $f(5) = 3$
Step 4: Substitute found values into the equation of a tangent line.	$f(x) - f(a) = f'(a)(x-a)$ $f(x) - 3 = \frac{1}{3}(x-5)$

b) Equation of the Normal Line.

Step 1: Find the slope of the normal line $\frac{-1}{\mathbf{f'(a)}}$.	Since $f'(5) = \frac{1}{3}$, then $\frac{-1}{\mathbf{f'(a)}} = -\mathbf{3}$
Step 2: Given the equation of a tangent line, swap slopes.	$f(x) - f(a) = \frac{-\mathbf{1}}{\mathbf{f'(a)}}(x-a)$ $f(x) - 3 = -3(x-5)$

Example 2: Find the equation of the tangent and normal lines of the function $f(x) = (x^2 - 1)^3$ at the point (2, 27).

Solution:

a) Equation of the Tangent Line.

Step 1: Find the slope of the function by solving for its first derivative.	$f(x) = (x^2-1)^3$ $f'(x) = 3(x^2-1)^2(2x)$ $f'(x) = 6x(x^2-1)^2$
Step 2: Knowing $f'(a)$, solve for the slope of the tangent at $a = 2$.	$f'(2) = 6(2)((2)^2-1)^2$ $f'(2) = 108$
Step 3: Solve for $f(a)$.	$f(2) = ((2)^2-1)^3$ $f(2) = 27$
Step 4: Substitute found values into the equation of a tangent line.	$f(x) - f(a) = f'(a)(x-a)$ $f(x) - 27 = 108(x-2)$

Tangents and Normals

b) Equation of the Normal Line.

Step 1: Find the slope of the normal line $\frac{-1}{\mathbf{f'(a)}}$.	Since $f'(2) = 108$, then $\frac{-1}{\mathbf{f'(a)}} = -\frac{1}{108}$
Step 2: Given the equation of a tangent line, swap slopes.	$f(x) - f(a) = \frac{-1}{\mathbf{f'(a)}}(x - a)$ $f(x) - 27 = -\frac{\mathbf{1}}{\mathbf{108}}(x - 2)$

<u>Exercises:</u>

1. Find the equation for the normal and tangent lines for f(x) at the specified points.

   a) f(x) = e^x at (0,1)
   b) f(x) = $2x^3 - 3x + 7$ at (1,6)
   c) f(x) = $\frac{1}{x^2}$ at (-1,1)
   d) f(x) = $x\cos(x)$ at (0,0)
   e) f(x) = xe^x at (0, 0)

<u>Solutions:</u>

1. a) Tangent: $y - 1 = x$, Normal: $y - 1 = -x$
   b) Tangent: $y - 6 = 3(x - 1)$, Normal: $y - 6 = -\frac{1}{3}(x - 1)$
   c) Tangent: $y - 1 = 2(x + 1)$, Normal: $y - 1 = -\frac{1}{2}(x + 1)$
   d) Tangent: $y = x$, Normal: $y = -x$
   e) Tangent: $y = x$, Normal: $y = -x$