Implicit Differentiation

Part A: Explicit versus Implicit Functions

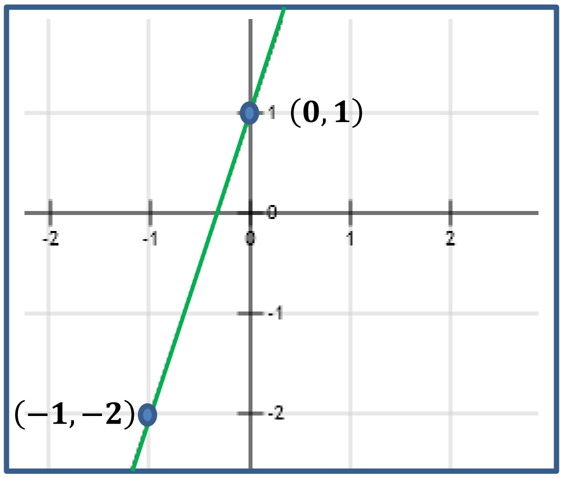
At this point, we have derived many functions, $\mathbf{y}$, written **EXPLICITLY** as functions of x.

What are explicit functions?

Given the function, $\mathbf{y = 3x + 1,}$

the value of $\mathbf{y}$ is dependent on the value of $\boldsymbol{x}$ (the independent variable). For every $\mathbf{x}$ value, we can easily find its corresponding $\mathbf{y}$ value by substituting and simplifying.

See **Figure 1:** $\mathbf{y = 3x + 1}$ below.



If $\boldsymbol{x}$ **= -1**, then $\mathbf{y = 3x + 1}$

$$= 3(-\mathbf{1}) + 1$$
$$= -2$$

Thus, our point on the graph is $(-1, -2)$.

If $\boldsymbol{x}$ **= 0**, then $\mathbf{y = 3x + 1}$

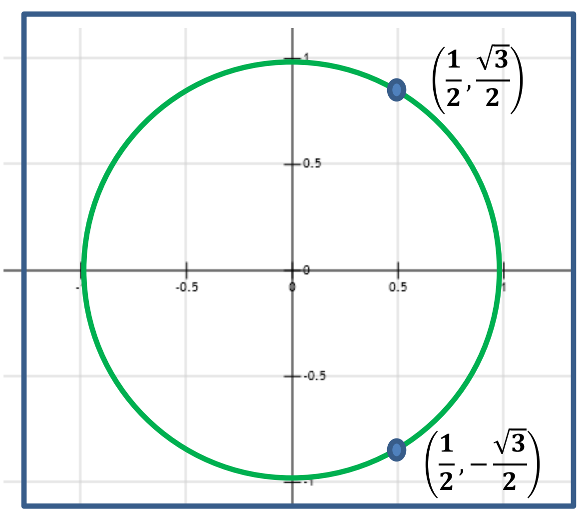
$$= 3(\mathbf{0}) + 1$$
$$= 1$$

Thus, our point on the graph is $(0, 1)$.

However, some functions, $\mathbf{y}$, are written **IMPLICITLY** as functions of x. In other words, the function is written in terms of $\mathbf{x}$ and $\mathbf{y}$. In these cases, we have to do some work to find the corresponding $\mathbf{y}$ value for each given $\mathbf{x}$.

An example of an *implicit function* includes, $\mathbf{x^2 + y^2 = 1.}$

See **Figure 2:** $\mathbf{x^2 + y^2 = 1}$ below.



If $\boldsymbol{x} = \frac{1}{2}$, then $\mathbf{x^2 + y^2 = 1}$

$$\left(\tfrac{1}{2}\right)^2 + y^2 = 1$$
$$y^2 = 1 - \left(\tfrac{1}{2}\right)^2$$
$$y = \pm\sqrt{1 - \left(\frac{1}{2}\right)^2}$$
$$y = \pm\sqrt{1 - \frac{1}{4}}$$
$$y = \pm\frac{\sqrt{3}}{2}$$

Thus, with a little bit of algebraic manipulation we find our points, $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.

Part B: Explicit Differentiation

Since explicit functions are given in terms of x, deriving with respect to x simply involves abiding by the rules for differentiation.

Example 1: Given the function, $\boldsymbol{y = 6x^2 + \frac{1}{2}(x^2 - 3)^4}$, find $y'\, or\ equivalently\ \frac{dy}{dx}$.

Step 1: Multiple both sides of the function by $\frac{d}{dx}$.	$\frac{\boldsymbol{d}}{\boldsymbol{dx}}(y) = \frac{\boldsymbol{d}}{\boldsymbol{dx}}\left(6x^2 + \frac{1}{2}(x^2 - 3)^4\right)$ $\frac{dy}{dx} = (\boldsymbol{6x^2})\frac{d}{dx} + \left(\frac{\boldsymbol{1}}{\boldsymbol{2}}(\boldsymbol{x^2 - 3})^4\right)\frac{d}{dx}$
Step 2: Differentiate both sides of the function with respect to x using the power and chain rule.	$\frac{dy}{dx} = (\boldsymbol{12x}) + (\boldsymbol{4x(x^2 - 3)^3})$

Part C: Implicit Differentiation

Method 1 – Step by Step using the Chain Rule

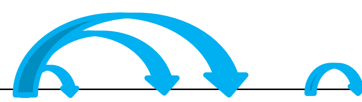
Since implicit functions are given in terms of $x\ and\ y$, deriving with respect to x involves the application of the chain rule.

Example 2: Given the function, $\mathbf{x^2 + y^2 = 1}$, find $y'\, or\ equivalently\ \frac{dy}{dx}$.

Step 1: Multiple both sides of the function by $\frac{d}{dx}$.	$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(1)$ $(x^2)\frac{\boldsymbol{d}}{\boldsymbol{dx}} + (y^2)\frac{\boldsymbol{d}}{\boldsymbol{dx}} = (1)\frac{\boldsymbol{d}}{\boldsymbol{dx}}$
Step 2: Differentiate $(x^2)\frac{\boldsymbol{d}}{\boldsymbol{dx}}$ $\boldsymbol{and}$ $(1)\frac{\boldsymbol{d}}{\boldsymbol{dx}}$ with respect to x.	$2x + (y^2)\frac{\boldsymbol{d}}{\boldsymbol{dx}} = 0$
Step 3: NOTE: We cannot differentiate $(y^2)\frac{\boldsymbol{d}}{\boldsymbol{dx}}$ with respect to $\boldsymbol{x}$ as above, since it	$2x + \boxed{(y^2)\frac{\boldsymbol{d}}{\boldsymbol{dx}}} = 0$

is written in terms of $\boldsymbol{y}$.	
Step 4: Perform the chain rule on $(\mathrm{y}^2)\frac{\boldsymbol{d}}{\boldsymbol{dx}}$ by 1. Differentiating y^2 with respect to y and 2. Multiplying the result by $\frac{dy}{dx}$ Remember, our goal is to solve for $\frac{dy}{dx}$.	$\frac{\boldsymbol{d}(\mathrm{y}^2)}{\boldsymbol{dx}} = \frac{\boldsymbol{d}(\mathrm{y}^2)}{\boldsymbol{dy}}\frac{\boldsymbol{dy}}{\boldsymbol{dx}}$ $\boxed{= \boldsymbol{2y}\,\frac{dy}{dx}}$
Step 5: Let $\frac{\boldsymbol{d}(\mathrm{y}^2)}{\boldsymbol{dx}} = \boldsymbol{2y}\,\frac{dy}{dx}$, and solve for $\frac{\boldsymbol{dy}}{\boldsymbol{dx}}$.	$2\mathrm{x} + (\mathrm{y}^2)\frac{\boldsymbol{d}}{\boldsymbol{dx}} = 0$ $2\mathrm{x} + \boldsymbol{2y}\,\frac{dy}{dx} = 0$ $\boldsymbol{2y}\,\frac{dy}{dx} = -2x$ $\frac{dy}{dx} = \frac{-2x}{2y}$ $\frac{\boldsymbol{dy}}{\boldsymbol{dx}} = \frac{-x}{y}$

Example 3: Given the function, $\boldsymbol{xy^3 + 4x^2 - 3x = 5}$, find $y'\, or\ equivalently\ \frac{dy}{dx}$.



Step 1: Multiple both sides of the function by $\frac{d}{dx}$.	$\frac{\boldsymbol{d}}{\boldsymbol{dx}}(xy^3 + 4x^2 - 3x) = \frac{\boldsymbol{d}}{\boldsymbol{dx}}(5)$ $(xy^3)\frac{\boldsymbol{d}}{\boldsymbol{dx}} + (4x^2)\frac{\boldsymbol{d}}{\boldsymbol{dx}} - (3x)\frac{\boldsymbol{d}}{\boldsymbol{dx}} = (5)\frac{\boldsymbol{d}}{\boldsymbol{dx}}$
Step 2: Differentiate $(4x^2)\frac{\boldsymbol{d}}{\boldsymbol{dx}}, (3x)\frac{\boldsymbol{d}}{\boldsymbol{dx}}, \boldsymbol{and}\ (5)\frac{\boldsymbol{d}}{\boldsymbol{dx}}$ with respect to x.	$(xy^3)\frac{\boldsymbol{d}}{\boldsymbol{dx}} + 8x - 3 = 0$
Step 3: NOTE: We cannot differentiate $(\mathrm{xy}^3)\frac{d}{dx}$ with respect to $\boldsymbol{x}$ alone, since $\boldsymbol{y}$ is in the term.	$\boxed{(xy^3)\frac{\boldsymbol{d}}{\boldsymbol{dx}}} + 8x - 3 = 0$

Step 4: Perform the product rule between the functions x and y^3 with respect to x.	$\frac{d(\mathrm{xy}^3)}{dx} = \left((x)\frac{d}{dx}\right)(y^3) + \left((y^3)\frac{d}{dx}\right)x$
Step 5: Where possible, derive with respect to x.	$= (1)(y^3) + x\left((y^3)\frac{d}{dx}\right)$
Step 6: Apply the chain rule to $(y^3)\frac{d}{dx}$ by 1. Differentiating y^3 with respect to y and 2. Multiplying the result by $\frac{dy}{dx}$ Remember, our goal is to solve for $\frac{dy}{dx}$.	$\left(\frac{d(y^3)}{dx}\right) = \frac{d(y^3)}{dy}\frac{dy}{dx}$ $= 3y^2\frac{dy}{dx}$
Step 7: Sub in our found values into $(xy^3)\frac{d}{dx} + 8x - 3 = 0$, and solve for $\frac{dy}{dx}$.	$[(\mathbf{xy}^3)\frac{\mathbf{d}}{\mathbf{dx}}] + 8x - 3 = 0$ $[(\mathbf{1})(y^3) + x\left((y^3)\frac{\mathbf{d}}{\mathbf{dx}}\right)] + 8x - 3 = 0$ $(y^3) + x\left(\mathbf{3y}^2\frac{\mathbf{dy}}{\mathbf{dx}}\right) + 8x - 3 = 0$ $y^3 + 3y^2x\frac{dy}{dx} + 8x - 3 = 0$ $\frac{dy}{dx} = \frac{-8x + 3 - y^3}{3y^2x}$

Method 2 – Chain Rule Short Cut

Once you have a good idea of how the chain rule works, you may begin to skip steps. In this method it is as though we are differentiating with respect to x and y at the same time with an added step. Every time you differentiate with respect to y, the term must be multiplied by a factor of $\frac{dy}{dx}$.

Example 4: Given the function, $4x^2 + \sin(xy^4) = 3y$, find $y'\,or\,equivalently\,\frac{dy}{dx}$.

Step 1: Differentiate both sides of the function with respect to x and y. Remember to multiply derivatives of y	$8x + \cos(xy^4)\,(1(y^4) + 4y^3x\frac{dy}{dx}) = 3\frac{dy}{dx}$

by a factor of $\frac{dy}{dx}$.	
Step 2: Expand.	$8x + \cos(xy^4)\,(1(y^4) + 4y^3x\frac{dy}{dx}) \; = 3\frac{dy}{dx}$ $8x + y^4\cos(xy^4) + 4y^3x\cos(xy^4)\frac{dy}{dx} \; = 3\frac{dy}{dx}$
Step 3: Collect and factor out $\frac{dy}{dx}$.	$4y^3x\cos(xy^4)\frac{dy}{dx} - \; 3\frac{dy}{dx} = -8x - y^4\cos(xy^4)$ $\frac{dy}{dx}(4y^3x\cos(xy^4) - \; 3) = -8x - y^4\cos(xy^4)$
Step 4: Solve for $\frac{dy}{dx}$**.**	$\frac{dy}{dx} = \frac{-8x - y^4\cos(xy^4)}{4y^3x\cos(xy^4) - \; 3}$

Example 5: Given the function, $xe^{x^2+y^2} = 5$, find $y'\, or\ equivalently\ \frac{dy}{dx}$.

Step 1: Apply the product rule between the functions $x\ and\ e^{x^2+y^2}$.	$(x)\frac{d}{dx}e^{x^2+y^2} + (e^{x^2+y^2})\frac{d}{dx}x = 5\frac{d}{dx}$
Step 2: Differentiate both sides of the function with respect to $\boldsymbol{x}$ and $\boldsymbol{y}$ at the same time. Remember to multiply derivatives of $\boldsymbol{y}$ by a factor of $\frac{dy}{dx}$.	$(1)e^{x^2+y^2} + (e^{x^2+y^2})(2x + 2y\frac{\boldsymbol{dy}}{\boldsymbol{dx}})\,x = 0$ $e^{x^2+y^2} + (e^{x^2+y^2})(2x^2 + 2yx\frac{\boldsymbol{dy}}{\boldsymbol{dx}}) = 0$
Step 3: Expand.	$e^{x^2+y^2} + (e^{x^2+y^2})(2x^2 + 2yx\frac{\boldsymbol{dy}}{\boldsymbol{dx}}) = 0$ $e^{x^2+y^2} + 2x^2e^{x^2+y^2} + 2yxe^{x^2+y^2}\frac{\boldsymbol{dy}}{\boldsymbol{dx}} = 0$
Step 4: Factor out $e^{x^2+y^2}$from the left hand side.	$e^{x^2+y^2}(1 + 2x^2) = \; -2yxe^{x^2+y^2}\frac{\boldsymbol{dy}}{\boldsymbol{dx}}$
Step 5: Solve for $\frac{dy}{dx}$**.**	$\frac{dy}{dx} = \frac{\mathbf{e}^{\mathbf{x}^2+\mathbf{y}^2}(1 + 2x^2)}{-2yx\mathbf{e}^{\mathbf{x}^2+\mathbf{y}^2}}$

Implicit Differentiation

	$\frac{dy}{dx} = \frac{-1-2x^2}{2yx}$

Exercises:

1. $x^3 + y^3 = 4$
2. $(x - y^2)^2 = 5x$
3. $y = x^2y + y^2x$
4. $cos(xy) + x^2 = -7y$
5. $y = \sin(2x + 3y)$
6. $e^{xy} = e^{cosy}$
7. $y - \ln y = 10x^3 - 6x^2 + 4$
8. $\tan y = \cos x$
9. $y - \sqrt{y} = \ln x$

Solutions:

1. $\frac{dy}{dx} = -\frac{x^2}{y^2}$
2. $\frac{dy}{dx} = \frac{5-2x+2y^2}{-4xy+4y^3}$
3. $\frac{dy}{dx} = \frac{2xy+y^2}{1-x^2-2yx}$
4. $\frac{dy}{dx} = \frac{ysin(xy)-2x}{7-xsin(xy)}$
5. $\frac{dy}{dx} = \frac{2cos(2x+3y)}{1-3cos(2x+2y)}$
6. $\frac{dy}{dx} = \frac{-ye^{xy}}{xe^{xy}+sinye^{cosy}}$
7. $\frac{dy}{dx} = \frac{30x^2y-12xy}{y-1}$
8. $\frac{dy}{dx} = -sinxcos^2$y
9. $\frac{dy}{dx} = \frac{2\sqrt{y}}{2x\sqrt{y}-x}$