Factoring Quadratic Expressions

Factoring the trinomial $ax^2 + bx + c$ when $a = 1$

A trinomial in the form $x^2 + bx + c$ can be factored to equal $(x + m)(x + n)$ when the product of $m \times n$ equals $c$ and the sum of $m + n$ equals $b$. (Note: the coefficient in front of $x^2$ must be 1)

**Step 1:** Common factor if you can.

**Step 2:** Find two integers (negative or positive whole numbers), $m$ and $n$, that multiply to equal $c$ (from $x^2 + bx + c$) AND add to equal $b$ (from $x^2 + bx + c$).

Start with finding integers that give you a product $c$ and then check which pair of numbers will add to equal $b$. Start with the product since there is a limited number of pairs that will give you the product $c$, while there is an infinite amount of numbers that can add to equal $b$.

**Step 3:** Substitute the numbers $m$ and $n$ directly into the expression $(x + m)(x + n)$

**Example 1:**

Factor. $x^2 + 7x + 12$

**Step 1:** Check to see if you can common factor first. In this case, there are no common factors for $x^2$, 7x and 12.

**Step 2:** Find two integers (negative or positive whole numbers), $m$ and $n$, that multiply to equal $c$ (from $x^2 + bx + c$) AND add to equal $b$ (from $x^2 + bx + c$).

$x^2 + 7x + 12$

$x^2 + bx + c$

$b = 7$

$c = 12$

$12 = 12 \times 1$

$12 = (-12) \times (-1)$

$12 = 6 \times 2$

$12 = (-6) \times (-2)$

$12 = 4 \times 3$

$12 = (-4) \times (-3)$

From the above pairs of integers only 3 and 4 add to 7.

**Step 3:** Substitute 3 and 4 into $(x + m)(x + n)$

$(x + m)(x + n)$

$(x + 3)(x + 4)$

Thus, $x^2 + 7x + 12 = (x + 3)(x + 4)$

**Step 3:** To double check that the factoring was done correctly, expand $(x + 3)(x + 4)$.

$(x + 3)(x + 4)$
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= \( x^2 + 4x + 3x + 12 \)
= \( x^2 + 7x + 12 \)

Example 2:
Factor. \( 2x^2 - 30x + 72 \)

Step 1: Common factor first, since \( 2x^2 \), \(-30x \) and \( 72 \) are all divisible by \( 2 \).
\( 2x^2 - 30x + 72 = 2(x^2 - 15x + 36) \)

Step 2: Now look at \( x^2 - 15x + 36 \) only and factor it.
\( x^2 - 15x + 36 \)
\( x^2 + bx + c \)
\( b = -15 \)
\( c = 36 \)

Find two integers (negative or positive whole numbers), \( m \) and \( n \), that multiply to equal \( 36 \) AND add to equal \( -15 \).

\[
\begin{align*}
36 &= 36 \times 1 \\
36 &= 18 \times 2 \\
36 &= 12 \times 3 \\
36 &= 9 \times 4 \\
36 &= 6 \times 6 \\
\end{align*}
\]

From the above pairs of integers only \( -12 \) and \( -3 \) add to \( -15 \).

Step 3: Substitute \( -12 \) and \( -3 \) into \((x + m)(x + n)\)
\[
(x + m)(x + n) = [x + (-12)] [x + (-3)]
= (x -12)(x - 3)
\]

Step 4: Put all of the factors together. Remember the common factor from Step 1.
\( x^2 - 15x + 36 = 2(x -12)(x - 3) \)

Practice Question:

1. Factor the following quadratic expressions:
   a) \( x^2 - 4x + 3 \)
   b) \( x^2 - 8x - 20 \)
   c) \( 5y^2 + 25y + 20 \)
   d) \( -3d^2 + 6d + 24 \)
   e) \( -2x^3 - 6x^2 + 56x \)
   f) \( 32 - 4x - x^2 \)

   Answers:
   1. a) \( (x - 1)(x - 3) \)
      b) \( (x - 10)(x + 2) \)
      c) \( 5(y + 4)(y + 1) \)
      d) \( -3(d - 4)(d + 2) \)
      e) \( -2(x + 7)(x - 4) \)
      f) \( -(x + 8)(x - 4) \)
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Factoring the trinomial $ax^2 + bx + c$ when $a \neq 1$

The decomposition method to factor a trinomial in the form $ax^2 + bx + c$ when $a$ does not equal 1 is presented below. (Note: there are other factoring methods).

**Step 1:** Check to see if you can common factor first.

**Decomposition Method:**

**Step 2:** Find two integers such that the product of these integers equals the product of $a$ and $c$, $ac$, (from $ax^2 + bx + c$) AND the sum of these integers equals $b$ (from $ax^2 + bx + c$).

**Step 3:** Use the two integers from Step 2 to re-write the middle term, $bx$, as the sum of these two integers.

**Step 4:** Common factor the first two terms of the algebraic expression. Then, common factor the last two terms of the algebraic expression. The objective of this step is to get two factors or brackets that are the same.

**Step 5:** Common factor the whole algebraic expression from Step 4.

**Example 1:**

**Factor.** $6x^2 + 13x – 5$.

**Step 1:** There is no common factor for 6, 13 and −5.

**Step 2:** Find two integers such that the product of these integers equals $ac$ and the sum equals $b$.

$6x^2 + 13x – 5$

$ax^2 + bx + c$

$ac = 6(−5) = −30$

$b = 13$

Start with looking for two integers whose product is −30 since there is an infinite number of pairs of integers whose sum is 13.

$5(−6) = −30 \quad −5(6) = −30$

$3(−10) = 30 \quad −3(10) = 30$

$15 (−2) = −30 \quad −15 (2) = −30$

$1(−30) = −30 \quad −1(30) = −30$

From the above pairs only $15 + (−2) = 13$.

**Step 3:** Use 15 and $−2$ to re-write the middle term, $13x$, as the sum of these two integers.

$13x = 15x + (−2x)$
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Therefore, \(6x^2 + 13x - 5 = 6x^2 + 15x - 2x - 5\)

**Step 4:** Common factor the first two terms of the algebraic expression, \(6x^2 + 15x\). Then, common factor the last two terms of the algebraic expression, \(-2x - 5\).

\[
6x^2 + 15x - 2x - 5
\]

\[
3x(2x + 5) - 1(2x + 5)
\]

Note: Factor out \(-1\), so the terms in the two brackets match.

**Step 5:** Common factor the resulting algebraic expression.

\[
3x(2x + 5) - 1(2x + 5)
\]

The common factor here is \((2x + 5)\) since both terms, \(3x(2x + 5)\) and \(-1(2x + 5)\) are divisible by \(2x + 5\).

\[
= (2x + 5)(3x - 1)
\]

Therefore, \(6x^2 + 13x - 5 = (2x + 5)(3x - 1)\)

**Step 6 (Optional):** To check the answer, expand the factored expression.

\[
(2x + 5)(3x - 1)
\]

\[
= 6x^2 - 2x + 15x - 5
\]

\[
= 6x^2 + 13x - 5
\]

**Example 2:**

**Factor.** \(-14x^2 + 116x - 32\).

**Step 1:** This trinomial can be common factored since \(-14, 116\) and \(-32\) are all divisible by \(-2\).

\[
-14x^2 + 116x - 32 = -2(7x^2 - 58x + 16)
\]

**Step 2:** Use decomposition method to factor the \(7x^2 - 58x + 16\) trinomial.

\[
ac = 7(16) = 112
\]

\[
b = -58
\]

Note: Since the product of the two integers is *positive*, but the sum is *negative*, both integers MUST be negative.
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−4(−28) = 112
−2(−56) = 112
(−14)(−8) = 112
(−7)(−16) = 112

From the above list, only −2+(-56) = −58

Step 3: Use −2 and −56 to re-write the middle term, −58x, as the sum of these two integers.

−58x = −2x + (−56x) = −2x − 56x

Therefore, 7x^2 − 58x + 16 = 7x^2 −2x −56x + 16

Step 4: Common factor the first two terms of the algebraic expression, 7x^2−2x. Then, common factor the last two terms of the algebraic expression, −56x + 16.

7x^2−2x −56x + 16

= x(7x −2) − 8(7x −2)

= x(7x −2) − 8(7x −2)

Factor out −8, not 8, since we want the brackets to be the same so that they become a common factor.

Step 5: Common factor the resulting algebraic expression.

x(7x −2) −8(7x −2) = (7x −2)(x − 8)

Step 6 (Optional): To check the answer, expand the factored expression.

(7x −2)(x − 8)

= 7x^2 − 56x − 2x + 16

= 7x^2 − 58x + 16

Practice Question:

2. Factor the following quadratic expressions. 
   a) 3g^2 + 2g − 8 
   b) 33y^2 + 5y − 2 
   c) 54x^2 + 12x −10 
   d) 9p^2 + 12p + 4 
   e) −30d^2 − 58d −16 
   f) 24x^2 − 6xy − 9y^2

Answers:

2. a) (3g − 4)(g + 2) 
   b) (3y + 1)(11y − 2) 
   c) 2(3x − 1)(9x + 5) 
   d) (3p + 2)^2 
   e) −2(5d + 8)(3d + 1) 
   f) 3(4x − 3y)(2x + y)