The Tangent Problem

Definition:

A **tangent line** is a line that "just touches" a curve at a specific point without intersecting it.

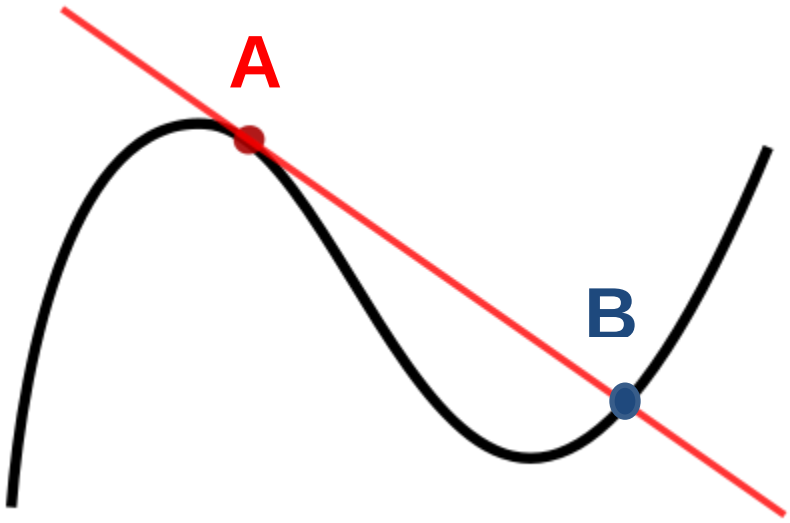


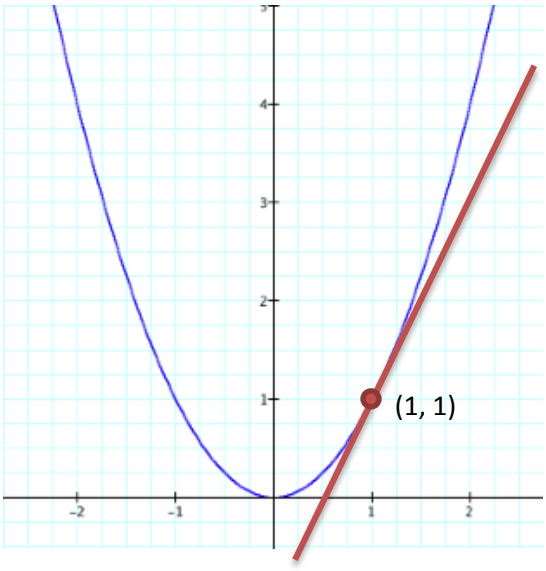
Figure 1: In this diagram, the line is tangent to point "A", but not to point "B."

Finding the equation of a tangent line to a curve at a specific point

In general, to find the equation of a line we need two points on the line **OR** the slope of the line and a single point.

Example 1: Find the equation of the tangent line for the function, $f(x) = x^2$ at $x = 1$.

Solution: Let's look at Figure 2 below: Graph of $f(x) = x^2$ and its tangent line at $x = 1$.



Recall, that the equation of a line is commonly written in the form, **y = mx + b**, where **m** is the slope and **b** is the y-intercept of a line.

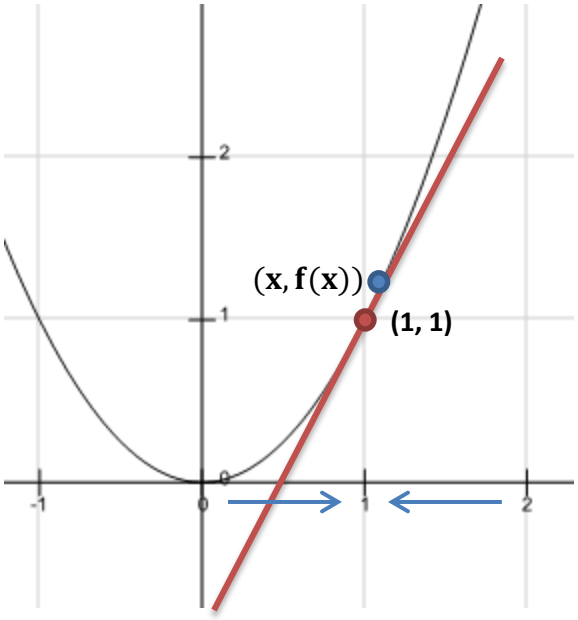
Graphically, we can *approximate* the equation of the tangent line at $\boldsymbol{x = 1}$. BUT this will not give us an accurate solution.

To find the equation of the tangent line at the point $\boldsymbol{x = 1}$, we use an algebraic approach. Thus, we either need to know two points on the tangent line or its slope and a point.

Recall: Slope is defined as rise over run, or the difference in $\mathbf{y}$ divided by the difference in $\boldsymbol{x}$.

$$slope = \frac{f(x) - f(a)}{x - a}$$

Looking at the question, we only know one point on the tangent line, (1, 1). Since we need at least two points to describe slope, we must choose another point, $(x, f(x))$, that is close to (1, 1) and belongs to the function, x^2. This way, the slope of the tangent line will be as accurate as possible.



Thus, our two points are (1, 1) and $(x, f(x)) = (x, x^2)$.

Let's sub these two points into our equation for slope. This gives us,

$$slope = \frac{f(x) - 1}{x - 1} = \frac{x^2 - 1}{x - 1}$$

for x really close to 1

The following table shows the slope of the tangent line for x approaching 1 from the left and right hand sides of the x-axis.

$\boldsymbol{x}$	**Slope**
2	3
1.5	2.5
1.1	2.1
1.01	2.01
1.001	*2.001*

$\boldsymbol{x}$	**Slope**
0.999	*1.999*
0.99	1.99
0.9	0.9
0.5	0.5
0	1

From the table above, it looks like the slope of the tangent line at the point (1, 1) is equal to 2. Thus, as x approaches 1 from the left and right hand sides, **m = 2**.

Given **m = 2** and the point **(1, 1)**, we can now find the y-intercept of the tangent line.

y = mx + b

$1 = (2)(1) + b$ Sub m = 2 and (1, 1) into y = mx + b

$1 = 2 + b$ Simplify

$1 - 2 = b$ Isolate for b

$b = -1$

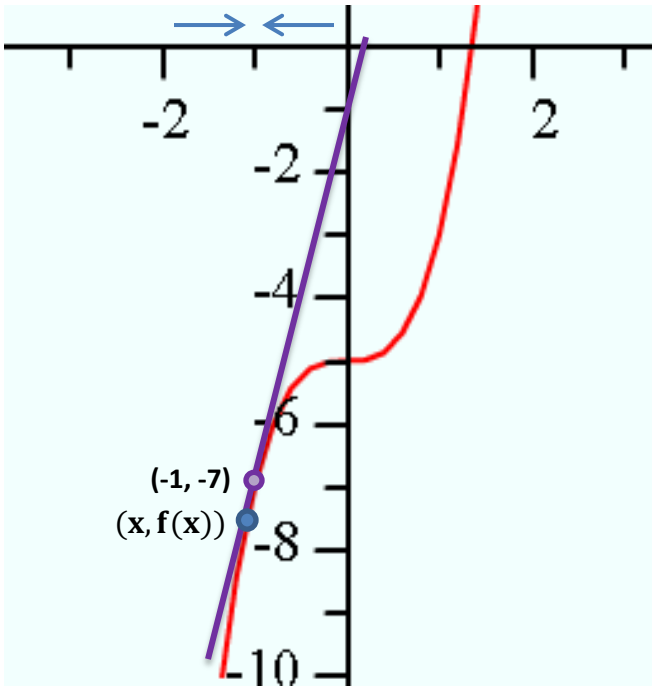
Therefore, the equation for the tangent line is $\mathbf{y = 2x - 1}$. Looking at Figure 2, we see that the equation of the tangent line matches with the graphical approximation.

In general, to find the slope of a tangent line to a curve at point (a, f(a)), the following formula can be used:

$$slope = \frac{f(x) - f(a)}{x - a} \text{ for } x \text{ really close to a}$$

Example 2: Find the equation of the tangent line for the function, $f(x) = 2x^3 - 5$, at $x = -1$.

Solution: To solve for the slope of the tangent line we need two points on the line.



Sub $x = -1$ into the function, $f(x) = 2x^3 - 5$ to solve for $f(-1)$.

$$f(-1) = 2(-1)^3 - 5 = -7$$

Thus, our first point and our point of tangency is, (-1, -7).

Now, we must choose another point, $(x, f(x))$, that is close to (-1, -7) and belongs to the function, $f(x)$.

Thus, our two points are (-1, -7) and $(x, f(x)) = (x,\ 2x^3 - 5)$. And our equation for slope becomes,

$$slope = \frac{f(x)-(-7)}{x-(-1)} = \frac{(2x^3-5)+7}{x+1} = \frac{2x^3+2}{x+1} \text{ for } x \text{ really close to } -1$$

The following table shows the slope of the tangent line as x approaches -1.

x	Slope	x	Slope
-2	14	-0.999	5.994002
-1.5	9.5	-0.99	5.9402
-1.1	6.62	-0.9	5.42
-1.01	6.0602	-0.5	3.5
-1.001	6.006002	0	2

From the table above, it looks like the slope of the tangent line is equal to 6. Thus, as x approaches -1 from the left and right hand sides, m = 6.

Given m = 6 and the point (-1, -7), we can now find the y-intercept of the tangent line.

y = mx + b

$-7 = (6)(-1) + b$ Sub in **m = 6** and **(-1, -7)** into y = mx + b

$-7 = -6 + b$ Simplify

$-7 + 6 = b$ Isolate for b

$b = -1$

Therefore, the equation for the tangent line is $\mathbf{y = 6x - 1}$.

Exercises:

Find the equation of the tangent line at the specified points.

1. $f(x) = x^3 \; at \; x = 2$
2. $f(x) = x^2 - 3x - 5 \; at \; x = 4$
3. $f(x) = -2x^3 + 8x^2 - 2 \; at \; x = 5$
4. $f(x) = 2(x-2)^3 + 2 \; at \; x = 3$

Solutions:

1. $y = 12x - 16$
2. $y = 5x - 21$
3. $y = -70x + 298$
4. $y = 6x - 16$