Continuity

In broad terms, a **continuous function** is a function that has a “smooth,” unbroken curve at every point on the interval. The graph of a **discontinuous function** is “broken” at one or more points. The graphs below illustrate these points.

![Continuous Function](image1)

**Figure 1:** A “smooth” continuous function at every point.

In more formal terms, a function \( f(x) \) is said to be continuous at a point \( x = a \) if

\[
\lim_{x \to a} f(x) = f(a)
\]

**Figure 2:** A discontinuous function at \( x = 0 \).

Given the definition of continuity, the following must be true:

1. \( \lim_{x \to a} f(x) \) exists
2. \( f(a) \) exists
3. \( \lim_{x \to a} f(x) \) is equal to \( f(a) \)

**Note:** The definition of continuity allows us to directly plug in the point “\( a \)” into a continuous function to evaluate the limit.

A point on a function may be discontinuous in the following three ways:

1. Infinite discontinuity
2. Removable discontinuity
3. Jump discontinuity
1. Infinite Discontinuity

**Figure 2** above is an example of an infinite discontinuity at the point \( x = 0 \). In this case we have a vertical asymptote at \( x = 0 \) and evaluating the limit at that point would result in \( \infty \) or \( -\infty \). Since the limit does not exist at \( x = 0 \), the function is not continuous at that point.

2. Removable Discontinuity

A **removable discontinuity** is when there is a hole on the curve of a function.

![Removable Discontinuity](image)

**Figure 2**: Removable discontinuity at \( x=1 \)

In **Figure 3**, we see a hole at the point \( x = 1 \). Evaluating the limit from the left and right hand side of \( x = 1 \), we have

\[
\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = \lim_{x \to 1} f(x) = 1
\]

However, evaluating \( f(1) \) we notice \( f(1) = 2 \).

Since, \( \lim_{x \to 1} f(x) \neq f(1) \) the function is not continuous at the point \( x = 1 \).

3. Jump Discontinuity

As the name implies, a **jump discontinuity** is when we have a “jump” in the function.
In Figure 4, we can see that there is a break in the function at the point $x = -1$. Since the left-hand side and right-hand side limits do not equal as $x$ approaches -1, the limit does not exist at that point. Thus, the function is discontinuous at the point $x = -1$.

Exercises:

In each case, determine where the function is discontinuous and identify the type of discontinuity.

a)
Solutions:

a) Discontinuous at x=0, Infinite discontinuity
   Discontinuous at x=3, Removable discontinuity

b) Discontinuous at x=0, Removable discontinuity
   Discontinuous at x=3, Jump discontinuity

c) Discontinuous at x=-3, Jump discontinuity
   Discontinuous at x=2, Removable discontinuity